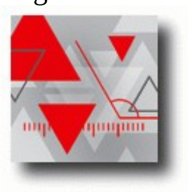
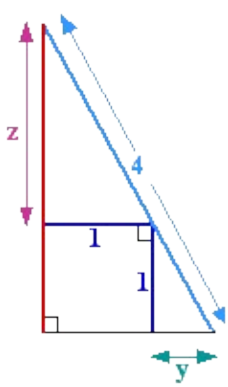

Mar 2002
Regulars



Puzzle page

We received quite a few correct solutions for this month's puzzle. In case it defeated you, the key was to look for symmetries in the information you are given, and to use *similar triangles.*

Part I



With y and z as in the diagram, $x = z + 1$, so it is sufficient to find z. By similar triangles, $z : 1$ as $1 : y$, so $zy = 1$.

By Pythagorasâ€™ Theorem,

$$\begin{aligned}(z+1)^2 + (y+1)^2 &= 16,\\ z^2 + 2z + 1 + y^2 + 2y + 1 &= 16,\end{aligned}$$

$$(z+y)^2 - 2zy + 2z + 2y + 2 = 16.$$

But $zy = 1$, so

$$(z+y)^2 + 2(z+y) - 16 = 0.$$

Solving this equation for $z+y$, and taking only the positive solution, yields

$$z + y = \sqrt{17} - 1.$$

Now we find $z - y$.

$$(z+y)^2 - (z-y)^2 = 4zy = 4,$$

$$\begin{aligned}
(z-y)^2 &= (z+y)^2 - 4 \\
&= (\sqrt{17}-1)^2 - 4 \\
&= 17 - 2\sqrt{17} + 1 - 4 \\
&= 14 - 2\sqrt{17}.
\end{aligned}$$

So

$$z + y = \sqrt{14 - 2\sqrt{17}}$$

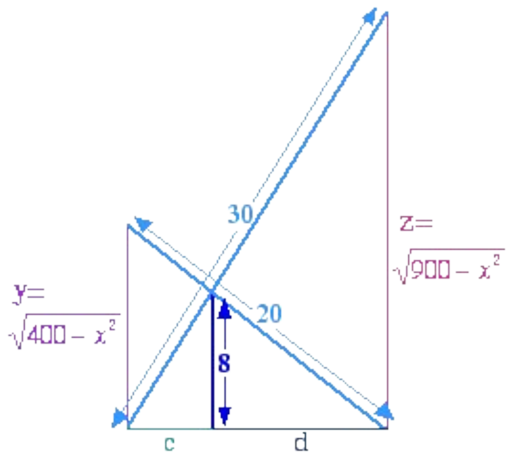
and

$$\begin{aligned}
2z &= (z+y) - (z-y) \\
&= (\sqrt{17}-1) - (\sqrt{14-2\sqrt{17}}) \\
&= 5.52.
\end{aligned}$$

Therefore

$$x = z + 1 = 2.76 + 1 = 3.76.$$

Part II



There are a number of approaches to this problem, but the one presented here follows from the observation that the information given is left−right symmetric − in other words, finding the two vertical heights will be equally easy (or hard!). We will label these two heights y and z, as in the diagram.

By similar triangles, and dividing x up into two segments c and d, we see that $y:x$ as $8:d$ and $z:x$ as $8:c$; therefore

$z = 8x/c \text{ and} y = 8x/d.$

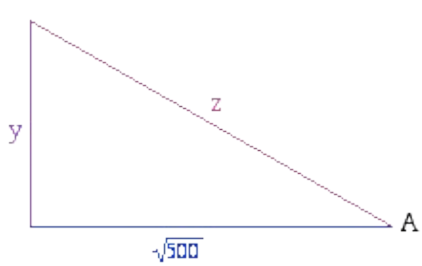
Adding gives $y+z = 8x^2/cd$ and multiplying gives $yz = 64x^2/cd.$

Combining these two results gives

$$\begin{aligned} yz &= 8(y+z); \\ (y+z)/yz &= 1/8; \\ 1/z + 1/y &= 1/8. \end{aligned}$$

Leaving this equation to one side for a moment, we can use Pythagorasâ€™ Theorem to find each of y and z in terms of our unknown x, as shown in the diagram above. Combining these two expressions gives that $z^2 - y^2 = 500$.

Using Pythagoras' Theorem, we draw a triangle to represent this relationship:



On its own, the information in this diagram is not enough to solve the triangle − for that, we would need one more bit of information. But now we can use the identity $yz = 8(y+z)$. All we have to do is find some combination of trig functions of A that takes the form $1/z + 1/y$. Since $cos(A) = \sqrt{500}/y$ and

$tan(A) = y/\sqrt{500}$, it is easy to see that

$$cos(A) + 1/tan(A) = \sqrt{500}(1/z + 1/y) = \sqrt{500}/8 = 2.7951.$$

Using a calculator gives $A = 27^o 38' 30''$ and $z = \sqrt{500}/cosA = 25.24$. Therefore

$$x = \sqrt{900 - (25.24)^2} = 16.2.$$

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.