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Regulars



Outer space: Relationships

by John D. Barrow



Most magazines have endless articles and correspondence about relationships and you will be pleased to hear that *Plus* is now no different. Why? Relationships are complicated and can often appear unpredictable and this is just the type of situation that mathematics can help you with. We even have a problem page.

The simplest relationships between things have a property that we call "transitivity" and it makes life simple. Being "taller than" is one of these transitive relationships. So if Ali is taller than Bob and Bob is taller than Carla, then Ali is necessarily taller than Carla. This relationship is in the nature of height. But not all relations are like this. Ali might like Bob and Bob might like Carla but that does not mean that Ali likes Carla. These "intransitive" relations can create very unusual situations when it comes to deciding what you should do when everyone does not agree.

But even when a relationship is transitive, things can get tricky when a group tries to come to an agreement. Suppose that Ali, Bob, and Carla decide to invest together in a secondhand car and go out to look at three different possibilities: Audi, BMW or Reliant Robin. They don't all agree on what to buy, so they decide that the outcome must be decided democratically. They must vote on it. So, each of them writes down their order of preference for the three makes:

	First choice	Second choice	Third choice
Ali	Audi	BMW	Reliant Robin
Bob	BMW	Reliant Robin	Audi
Carla	Reliant Robin	Audi	BMW

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Each of the friends separately describes their – transitive – preferences, but something odd happens when they add up their votes. At first, things look promising: Audi beats BMW by 2 preferences to 1, and BMW beats the Reliant Robin by 2 preferences to 1. But, strangely, the Reliant Robin beats the Audi by 2 preferences to 1. Although preference, unlike "liking", is a transitive relationship – if you prefer A to B and B to C, then you prefer A to C – it can create awkward paradoxes when a group tries to decide on an overall preference. Small elections to decide who you prefer amongst candidates for a job, who captains a sports team, or even what car to buy, are fraught with paradox. Let the voter beware.



Should we bother tidying up? Let's vote!

Faced with this trilemma, Ali, Bob and Carla decided to give up on the car-buying and put their resources into renting a house together. Alas, more decisions were soon necessary. Should they decorate the living room? Should they tidy the garden? Should they buy a new TV? There was no consensus so they decided to vote "yes" or "no" on each of the three issues in turn. Here's what they said:

	Decorate house?	Tidy garden?	Buy TV?
Ali	Yes	Yes	No
Bob	No	Yes	Yes
Carla	Yes	No	Yes
Majority decision	YES	YES	YES

All seemed clear. There was a majority of two-to-one to do all three things.

But then money seemed to be running short and the trio realised that they needed two more people to share the house if they were to pay the rent. After just a few phone calls they had found new house-mates Dell and Tracy, who rapidly moved in all their belongings. Of course, they thought that it was only fair they be included in the household vote on the decoration, gardening and TV purchase. They both voted "No" to each of the three proposals while Ali, Bob and Carla stuck to their earlier decisions.

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Perhaps you can show that a very strange situation has now been created in the enlarged household: *a majority of the household are on the losing side in the vote on a majority of the decisions!*



Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.