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Regulars

Editorial

Election issues

Whenever major elections come around public attention swings, albeit briefly, to a mathematical aspect of democracy: how to devise a voting system that reflects the true "will of the people". The arguments against the systems used in the US presidential elections and the UK general elections are well−rehearsed: their reliance on electoral bodies like the Electoral College in the US and MPs in the UK, means that the winner does not necessarily hold an overall majority, and the first−past−the−post feature can split majorities and invites, even forces, strategic voting that can distort the people's real preferences beyond recognition.

The Condorcet paradox

Suppose that there are three candidates and three hundred voters. The voters split into three groups with the order of preference of each group as follows:

Group 1, 100 voters	ABC
Group 2, 100 voters	BCA
Group 3, 100 voters	CAB

Whichever candidate is chosen as the winner, two thirds of the electorate is unhappy because they preferred someone else.

These, you might think, are anomalies of two particularly crude systems. What few people know, however, is that voting paradoxes are mathematically inevitable, no matter which system you choose. One situation that no voting system can resolve easily is known as *Condorcet's paradox*. Here voters' preferences divide up cyclically, so that, no matter which candidate is chosen as the winner, a majority of people are left unhappy, because they preferred another candidate. What's more, if the candidates were to fight it out in pairwise contests, the outcome would depend crucially on the order in which the contests took place.

A Condorcet winner

In this election with three candidates and 65 voters, B beats both A and C in pairwise contests: A total of 35 voters prefer B to A, while only 30 prefer A to B, and a total of 45 voters prefer B to C, while only 30 prefer C to B. Candidate B is therefore the unique Condorcet winner. In a first past the post system, however, B would come last.

Group 1, 30 voters	ABC
Group 2, 15 voters	BCA
Group 3, 20 voters	CBA

Such a symmetric split of preferences is unrealistic, you might say, so let's look at the other extreme when there is one candidate who beats *all* others in pairwise contests. If such a *Condorcet winner* exists, then surely he or she will win the election under any reasonable system? The answer, maybe surprisingly, is no: Condorcet winners do not necessarily hold an overall majority, so first past the post systems, and even those that take voters second or third preferences into account, are not guaranteed to elect a Condorcet winner.

What a voting system has to do is somehow transform what the voters have in their heads a ranking of candidates in order of preference into a single final ranking that makes reasonable democratic sense. You can set yourself some obvious guidelines here, for example that all voters' rankings should carry equal weight (so you don't end up with a dictator), and that the system should be deterministic in the sense that the same election run twice gives the same outcome. This is where the most interesting voting paradox kicks in: *Arrow's impossibility theorem* says that if voters are faced with more than two candidates, then a set of very reasonable conditions, including the two just mentioned, become incompatible. Finding a final ranking that sticks to all of these, pretty basic, conditions is a mathematical impossibility something, somewhere has to give.

So has mathematics just wiped out our chance of democracy? Not at all. Arrow's theorem doesn't say that every voting system is horribly flawed, rather it says that for every voting system there is the theoretical possibility of an anomaly resulting from voters splitting up along certain lines. It still makes a lot of sense to try and improve voting systems. The real threat to democracy comes, as ever, not from mathematical theorems, but from people.

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